New perspectives on credible intervals and sample size determination

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Overview

- Notations
- Credible intervals
  - Equal-tails intervals (ETs)
  - Highest density intervals (HDs)
- Traditional SSD criteria
  - length criteria
  - location criteria
- New SSD criteria
  - A – Based on discrepancy between alternative exact intervals
  - B – Based on discrepancy between approximate and exact intervals

Notations

- parametric model

$$\{f_n(\cdot;\theta), \quad \theta \in \Omega\}$$

- data

$$\mathbf{x}_n = (x_1, \cdots, x_n)$$

- prior distribution

$$\pi(\theta)$$

- posterior distribution

$$\pi(\theta|\mathbf{x}_n) = \frac{f_n(\mathbf{x}_n|\theta)\pi(\theta)}{\int_\Omega f_n(\mathbf{x}_n|\theta)\pi(\theta)d\theta}$$

Parameter

θ represents

- unknown effect of a treatment
- unkwown difference between effects of treatments
- ...

Θ is a random variable

Credible interval

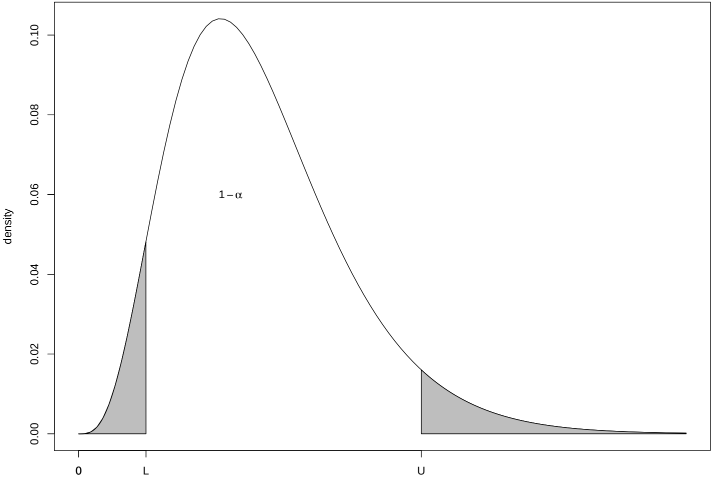
A $(1-\gamma)$-credible interval

$$C(\mathbf{x}_n) = [\ell(\mathbf{x}_n), u(\mathbf{x}_n)]$$

- is subset of Ω (parameter space)
- depends on $\mathbf{x}_n$ and $\pi(\theta)$
- has posterior probability equal to $1-\gamma$

$$\mathbb{P}\left[\Theta \in C | \mathbf{x}_n\right] = \int_C \pi(\theta|\mathbf{x}_n) d\theta = 1-\gamma$$

Credible interval



ET and HD credible intervals

- Equal-tails intervals (ETs)

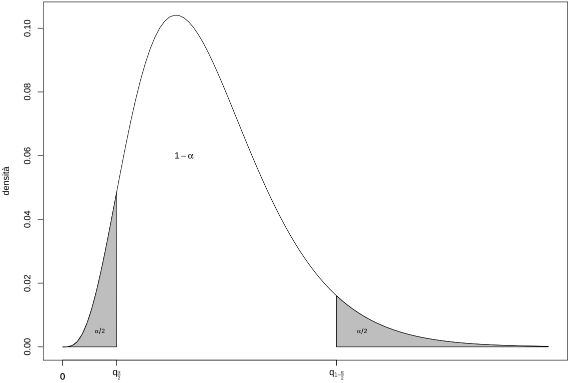
  $C_e(\mathbf{x}_n) = [\ell_e(\mathbf{x}_n), u_e(\mathbf{x}_n)]$

- Highest (posterior) density intervals (HDs)

  $C_h(\mathbf{x}_n) = [\ell_h(\mathbf{x}_n), u_h(\mathbf{x}_n)]$

Equal-tails intervals

$$\ell_e = q_{\frac{\gamma}{2}} \qquad u_e = q_{1-\frac{\gamma}{2}}$$



Equal-tails intervals

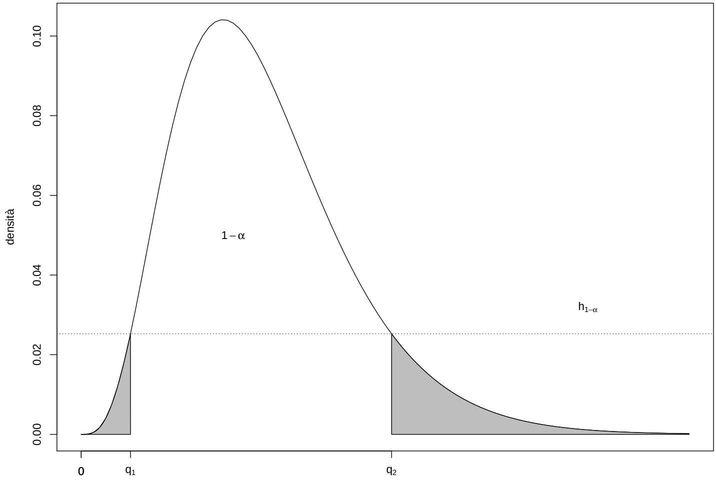
$C_e = [\ell_e, u_e]$

- $\mathbb{P}[\ell_e \leq \Theta \leq u_e | \mathbf{x}_n] = 1 - \gamma$
- $\mathbb{P}[\Theta \leq \ell_e | \mathbf{x}_n] = Pr[\Theta \geq u_e | \mathbf{x}_n] = \frac{\gamma}{2}$

Features

- easy-to-obtain
- invariant under transformation
- non-optimal in length

Highest posterior density intervals



Highest posterior density intervals

$C_h = [\ell_h, u_h]$

- $\mathbb{P}[\ell_e \leq \Theta \leq u_e | \mathbf{x}_n] = 1 - \gamma$

- $\pi(\theta | \mathbf{x}_n) \geq h_{1-\gamma}, \qquad \forall\, \theta \in C_h$

Features

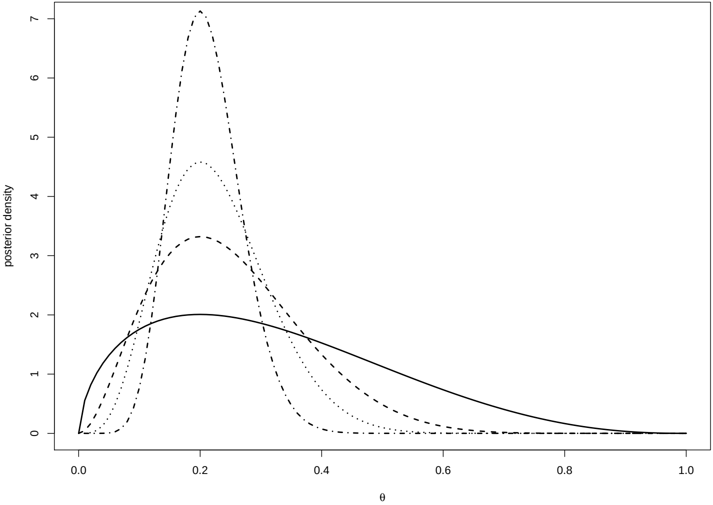
- not easy-to-obtain
- not invariant under transformation
- optimal in length

Credible intervals and sample size

As n increases, posterior density tends

- to concentrate around θ^*
- to symmetrize
- to normalize

Credible intervals and sample size



Credible intervals and sample size

As n increases, alternative credible intervals tend

- to become shorter and shorter
- to concentrate around θ^* (true parameter value)
- to coincide

Traditional Bayesian SSD criteria

Consider

$$C(\mathbf{X}_n) = [\ell(\mathbf{X}_n), u(\mathbf{X}_n)]$$

where $\mathbf{X}_n$ is random

Look at

- Length: want C to be short
- Location: want C either inside or outside a range of equivalence

Criteria based on *Length*

To control *precision* of intervals

Consider

$$C(\mathbf{X}_n) = [\ell(\mathbf{X}_n), u(\mathbf{X}_n)]$$

Define

$$\mathcal{L}(\mathbf{X}_n) = u(\mathbf{X}_n) - \ell(\mathbf{X}_n)$$

Determine

$$\min n \in \mathbb{N} \ : \ \mathbb{E}[\mathcal{L}(\mathbf{X}_n)] < \epsilon$$

Criteria based on *Location* (cont.)

To control the location of C w.r.t. a range of equivalence

$$\mathcal{I} = [\theta_i, \theta_s] \subset \Omega$$

- Superiority trials

$$\min n \in \mathbb{N} \; : \; \mathbb{E}[\ell(\mathbf{X}_n)] > \theta_s$$

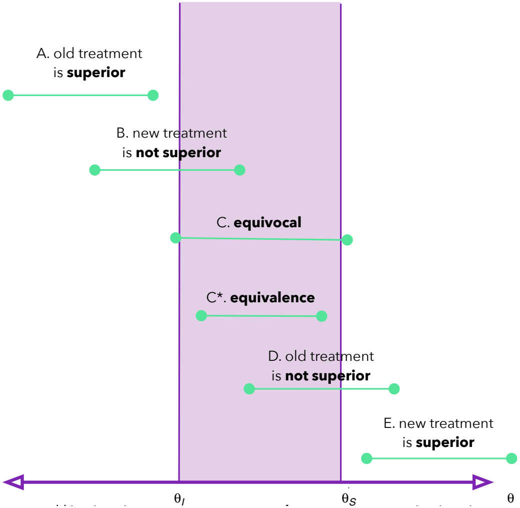
- Inferiority trials

$$\min n \in \mathbb{N} \; : \; \mathbb{E}[u(\mathbf{X}_n)] < \theta_i$$

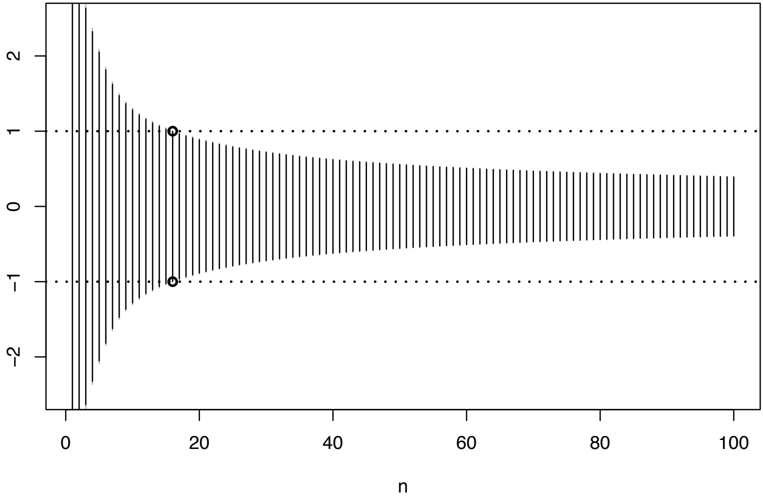
- Equivalence trials

$$\min n \in \mathbb{N} \; : \; \mathbb{E}[\ell(\mathbf{X}_n)] > \theta_s \qquad \text{and} \qquad \mathbb{E}[u(\mathbf{X}_n)] < \theta_i$$

Criteria based on *Location*



Example (equivalence)



Alternative approaches

Based on

consensus

A) between alternative exact intervals

- equal-tails intervals
- highest density intervals

B) between approximate and exact intervals

- intervals based on normal approximations
- highest density intervals and/ or equal-tails intervals

Motivations

Why do we want consensus?

A) want an exact interval with the good properties of both ETs and HDs

B) want an approximate interval that is

- easy to compute
- close to an exact credible interval

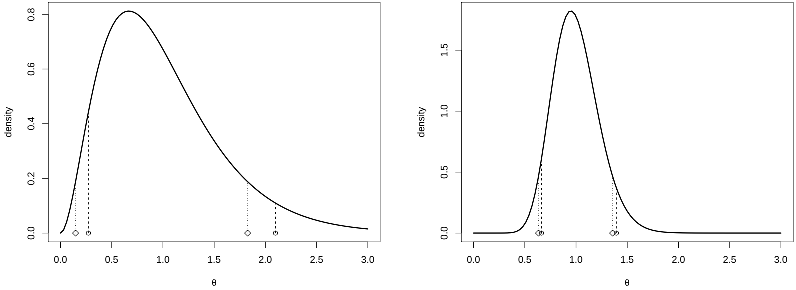
Part A – Consensus between ETs and HDs

Ideal

an interval with good properties of both ETs and HDs

Bad news - good news!

- small n: ETs and HDs do not coincide (in general)
- large n: closer and closer



Idea!

proceed as follows

- quantify the discrepancy between C_e and C_h
- consider the discrepancy as a random variable
- evaluate the expected value of the discrepancy
- select the smallest sample size such that the expected discrepancy is small

Discrepancy between C_e and C_h

more formally

- $D_n(\mathbf{x}_n) = d(C_e, C_h)$
- $D_n(\mathbf{X}_n)$ as a random variable
- $e_n^D = \mathbb{E}[D_n]$
- n^*, optimal sample size as

$$n^* = \min\{n \in \mathbb{N} : \; e_n^D \leq \epsilon\}$$

Types of discrepancies between intervals

Alternative measures D_n, based on

- distance between bounds of C_e and C_h
- difference in probabilities of tails between C_e and C_h

“Bounds” discrepancy

- $B_n = |\ell_e - \ell_h| + |u_e - u_h|$
- $e_n^B = \mathbb{E}[B_n]$
- $n^* = \{\min \mathbb{N} : e_n^B < \epsilon\}$

BUT: e_n^B is not a relative measure

“Tails” discrepancy

Idea

- note that as n increases C_h becomes an ET interval
- define a measure T_n such that is
  - equal to zero if C_h has tails-probability both equal to $\frac{\gamma}{2}$
  - greater than zero if the tails-probability of C_h are not equal to $\frac{\gamma}{2}$

"Tails" discrepancy

It can be easily shown that

$$T_n(\mathbf{x}_n) = \frac{|2F_n(\ell_h|\mathbf{x}_n) - \gamma|}{\gamma}$$

is a relative measure of discrepancy between C_h and the ET interval (where $F_n(\cdot|\mathbf{x}_n)$ is the posterior c.d.f)

"Tails" discrepancy and SSD

Therefore

$$n^* = \{\min \mathbb{N} : e_n^T < \epsilon\}$$

where

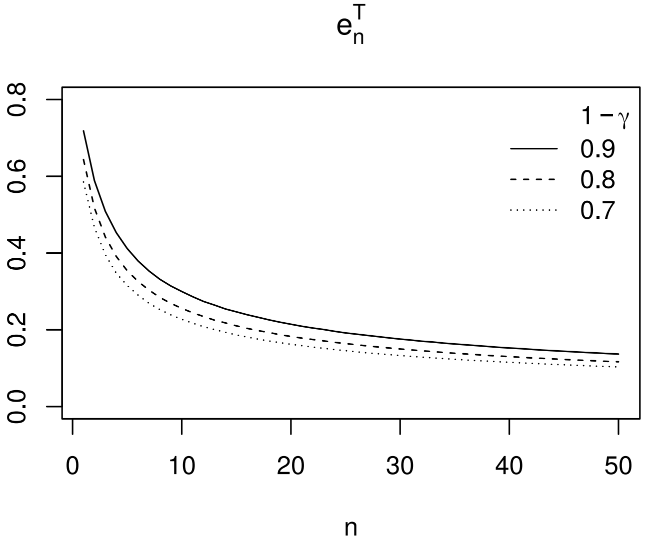
$$e_n^T = \mathbb{E}[T_n]$$

Main comments

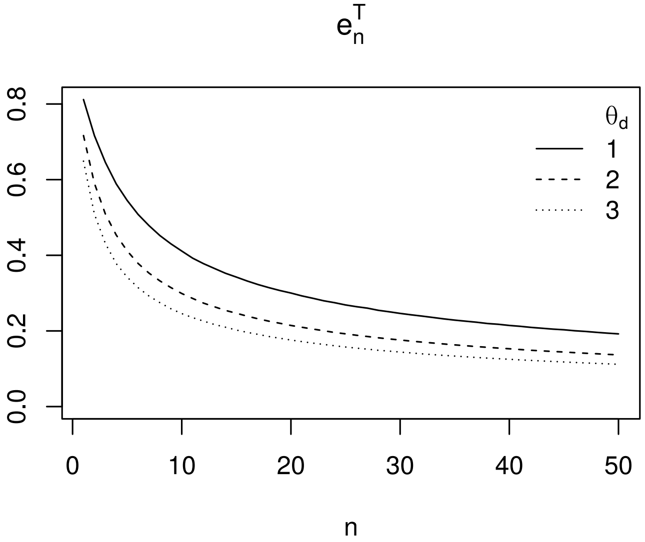
n^* increases with

- credibility level $1 - \gamma$
- asymmetry of the likelihood function
- asymmetry of the prior density

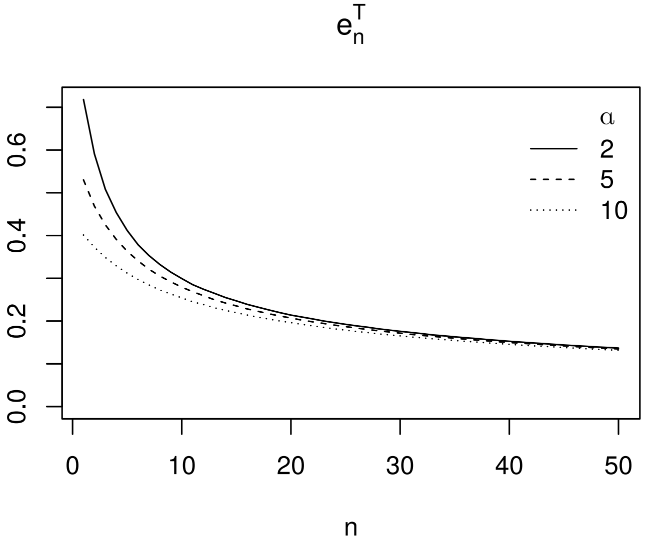
Example (Poisson model): effect of $1 - \gamma$



Example (Poisson model): effect of likelihood



Example (Poisson model): effect of prior



Example (Poisson model): relationships with skewness

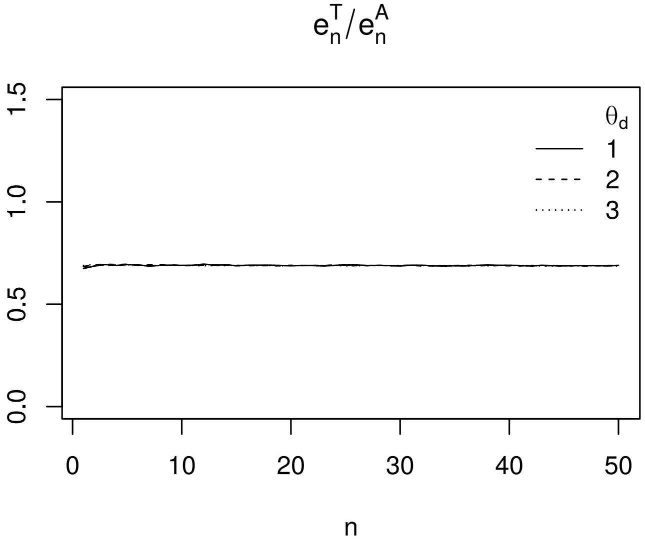
- $T_n(\mathbf{x})$ is a measure of skewness of $\pi(\theta|\mathbf{x}_n)$
- Equivalent to

$$A_n(\mathbf{x}_n) = \frac{\bar{\mu}_3(\mathbf{x}_n)}{\sigma^3(\mathbf{x}_n)}$$

- Consequence

$$\frac{e_n^T}{e_n^A} \simeq 1$$

Example (Poisson model): relationships with skewness



Example (Poisson model): optimal sample sizes

n^* using e_n^T

	$1-\gamma$	0.9			0.8			0.7		
	ϵ	0.15	0.2	0.25	0.15	0.2	0.25	0.15	0.2	0.25
θ_d	1	83	47	30	60	34	22	48	27	17
	2	42	24	15	30	17	11	24	14	9
	3	28	16	10	21	12	8	16	9	6

θ_d = design value of θ

Part B – Approximate and exact intervals

Let

$$\widetilde{C} = [\tilde{\ell}_n, \tilde{u}_n]$$

be an approximate credible interval.

For instance the likelihood interval

$$\widetilde{C} = [\hat{\theta} - z_{1-\frac{\alpha}{2}}\sqrt{I_n(\hat{\theta})^{-1}}, \hat{\theta} + z_{1-\frac{\alpha}{2}}\sqrt{I_n(\hat{\theta})^{-1}}]$$

Approximate intervals

Want minimum n such that $\widetilde{C}$ is approximately

- $(1-\gamma)$-credible interval
- ET interval
- HD interval

Measures of discrepancy for approximate intervals

Define 3 measures of discrepancy

$$P_n,\ T_n,\ B_n$$

such that

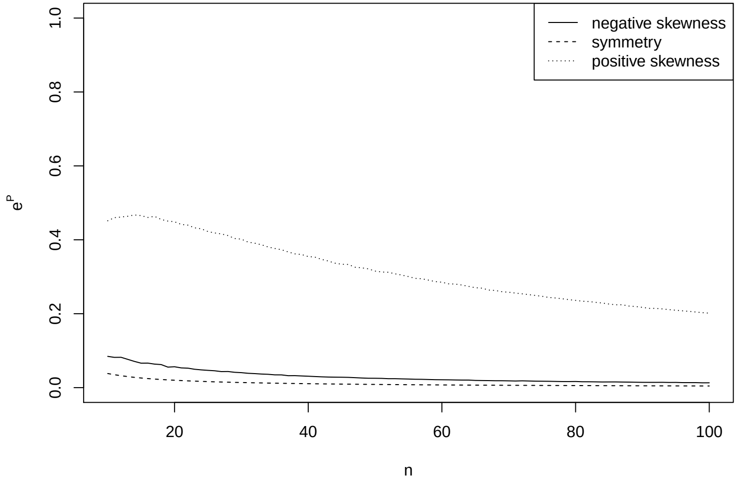
- small $P_n \Rightarrow \widetilde{C}$ is approximately a $(1-\gamma)$ CI
- small $T_n \Rightarrow \widetilde{C}$ is is approximately ET
- small $B_n \Rightarrow \widetilde{C}$ is is approximately HD

SSD for approximate intervals

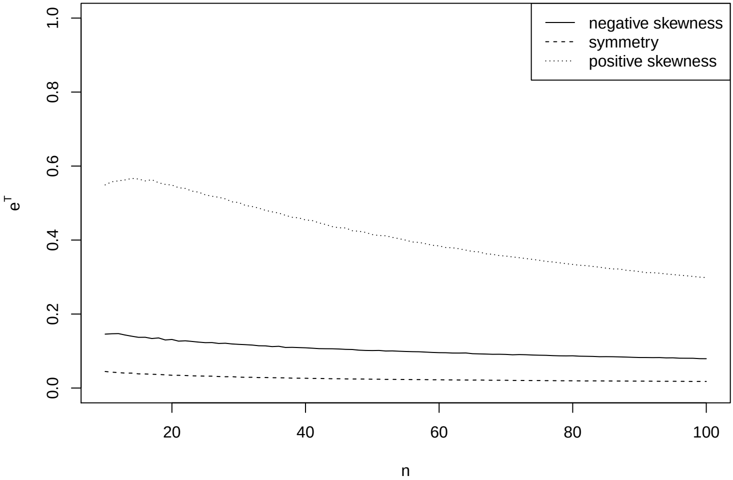
For SSD, consider

- $e_n^P = \mathbb{E}[P_n]$
- $e_n^T = \mathbb{E}[T_n]$
- $e_n^B = \mathbb{E}[B_n]$

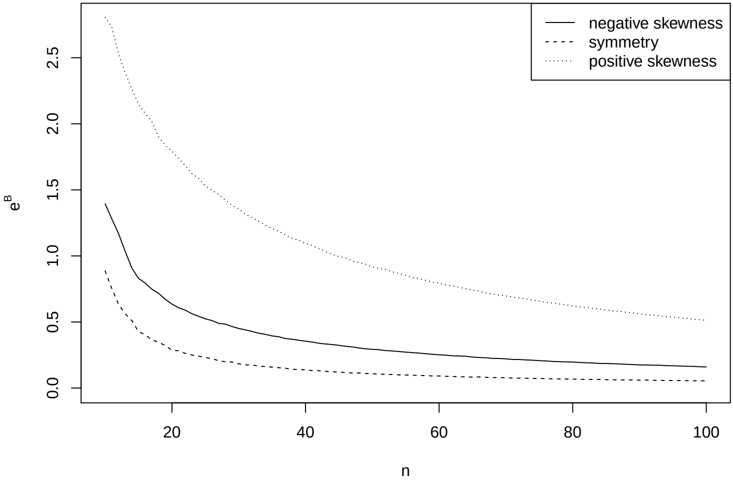
Example: approx CI for log-odds ($e_n^P = \mathbb{E}[P_n]$)



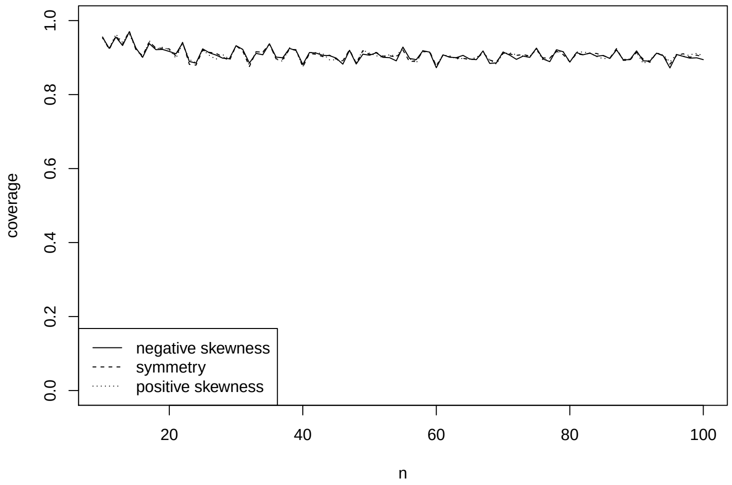
Example: approx CI for log-odds ($e_n^T = \mathbb{E}[T_n]$)



Example: approx CI for log-odds ($e_n^B = \mathbb{E}[B_n]$)



Example: approx CI for log-odds (coverage)



Final comments

1. Selection of measures of discrepancy
2. Difficulties in fixing some thresholds (B_n is not a relative measure)
3. Relationhips with other SSD criteria
4. Multiparametric problems

References: general

Cao J., Lee J. J. and Albert, S. (2009). Comparison of Bayesian sample size determination criteria: ACC, ALC, and WOC. *Journal of Statistical Planning and Inference*, 139, 4111 - 4122.

Joseph L. and Wolfson D. (1997). Interval-based versus decision theoretic criteria for the choice of sample size. *Journal of the Royal Statistical Society.* Series D (*The Statistician*), 46, 145 - 149.

References: range of equivalence

Brutti P. and De Santis F. (2008). Robust Bayesian sample size determination for avoiding the range of equivalence in clinical trials. *Journal of Statistical Planning and Inference*, 138, 1577 - 1591.

Gubbiotti S. and De Santis, F. (2011). A Bayesian method for the choice of the sample size in equivalence trials. *Australian and New Zealand Journal of Statistics*, 53, 443 - 460.

References: control of discrepancy

De Santis F. and Gubbiotti S. (2019). Progressive overlap of equal-tails and highest posterior density intervals. Submitted.

De Santis F. and Gubbiotti S. (2019). Sample size determination for calibrated approximate credible intervals in clinical trials. In progress.

"Tails" discrepancy

Idea: check when the HD interval becomes an ET interval

- $C_h = [\ell_h, u_h]$
- $F_n(\cdot|\mathbf{x}_n)$ (posterior c.d.f)
- $F_n(\ell_h|\mathbf{x}_n)$ and $1 - F_n(u_h|\mathbf{x}_n)$
- $T_1 = |F_n(\ell_h|\mathbf{x}_n) - \gamma/2|$ and $T_2 = |1 - F_n(u_h|\mathbf{x}_n) - \gamma/2|$

"Tails" discrepancy (cont.)

Note that

- $T_1 + T_2 = |2F_n(\ell_e|\mathbf{x}_n) - \gamma|$
- $0 \leq |2F_n(\ell_e|\mathbf{x}_n) - \gamma| \leq \gamma$

Therefore a relative measure of discrepancy is

$$T_n(\mathbf{x}_n) = \frac{|2F_n(\ell_h|\mathbf{x}_n) - \gamma|}{\gamma}$$

“Tails” discrepancy (cont.)

- $T_n(\mathbf{x}_n) = \frac{|2F_n(\ell_h|\mathbf{x}_n) - \gamma|}{\gamma}$
- $e_n^T = \mathbb{E}[T_n]$
- $n^* = \{\min \mathbb{N} : e_n^T < \epsilon\}$

Measures of discrepancy for approximate intervals

$\widetilde{C}(\mathbf{x}_n) = [\tilde{\ell}_n, \tilde{u}_n]$ is

approximately $1-\gamma$ if

$$P_n = |F_n(\tilde{u}_n) - F_n(\tilde{\ell}_n) - (1-\gamma)| \qquad \text{small}$$

approximately ET if

$$T_n = |F_n(\tilde{u}_n) - \frac{\gamma}{2}| + |1 - F_n(\tilde{\ell}_n) - \frac{\gamma}{2}| \qquad \text{small}$$

approximately HD if

$$B_n = |\tilde{\ell}_n - \ell_h| + |\tilde{u}_n - u_h|$$